THE STRUCTURE OF GLOBAL ATTRACTORS FOR DISSIPATIVE ZAKHAROV SYSTEMS WITH FORCING ON THE TORUS

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ABSTRACT. The Zakharov system was originally proposed to study the propagation of Langmuir waves in an ionized plasma. In this paper, motivated by the work of the first and third authors in [5], we numerically and analytically investigate the dynamics of the dissipative Zakharov system on the torus in 1 dimension. We find an interesting family of stable periodic orbits and fixed points, and explore bifurcations of those points as we take weaker and weaker dissipation.

1. INTRODUCTION

In this paper we study the dissipative Zakharov system with forcing:

$$
(1)\qquad \begin{cases} iu_t + u_{xx} + i\gamma u = nu + f, \quad x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}), \quad t \in [0, \infty), \\ n_{tt} - n_{xx} + \delta n_t = (|u|^2)_{xx}, \\ u(x, 0) = u_0(x) \in H^1(\mathbb{T}), \\ n(x, 0) = n_0(x) \in L^2(\mathbb{T}), \quad n_t(x, 0) = n_1(x) \in H^{-1}(\mathbb{T}), \quad f \in H^1(\mathbb{T}). \end{cases}
$$

The original Zakharov system ($\gamma = \delta = f = 0$) was proposed in [14] as a model for the collapse of Langmuir waves in an ionized plasma. The complex valued function $u(x,t)$ denotes the slowly varying envelope of the electric field with a prescribed frequency and the real valued function $n(x,t)$ denotes the deviation of the ion density from the equilibrium. Smooth solutions of the Zakharov system obey the following conservation laws:

$$\|u(t)\|_{L^2(\mathbb{T})} = \|u_0\|_{L^2(\mathbb{T})}$$

and

$$E(u, n, \nu)(t) = \int_{\mathbb{T}} |\partial_x u|^2 dx + \frac{1}{2}\int_{\mathbb{T}} n^2 dx + \frac{1}{2}\int_{\mathbb{T}} \nu^2 dx + \int_{\mathbb{T}} n|u|^2 dx = E(u_0, n_0, n_1)$$

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where ν is such that $n_t = \nu_x$ and $\nu_t = (n + |u|^2)_x$. These conservation laws identify $H^1 \times L^2 \times H^{-1}$ as the natural energy space for the system. Local and global well-posedness in the energy space was established by Bourgain [2]. Lower regularity optimal results were obtained by Takaoka in [11]. The well-posedness theory extends to the dissipative and forced system without difficulty [5].

In [5], the first and third authors established a smoothing property for the Zakharov system, and as a corollary they proved the existence and smoothness of a global attractor in the energy space. For a discussion of basic facts about global attractors see [12] and [5]. The problem with Dirichlet boundary conditions had been considered in [6] and [7] in more regular spaces than the energy space. The regularity of the attractor in Gevrey spaces with periodic boundary conditions was considered in [10].

Here, we primarily focus on the dynamics of solutions to (1). For large dissipation we prove that the global attractor is a single point consisting of a unique stable stationary solution of the system. Then, we proceed to investigate numerically the case of smaller dissipation in the spirit of the numerical exploration of damped-forced Korteweg-de Vries equation in [3] and for the Waveguide Array Mode-Locking Model in [13]. In particular, we explore equilibrium and periodic solutions, the branching of solutions, bifurcation points, period doubling and other interesting dynamical structures that arise.

The paper is organized as follows. In Section 2, we obtain preliminary estimates on the solutions and study the existence and uniqueness of stationary solutions. In Section 3, we prove that in the case of large dissipation, the global attractor consists of the unique stationary solution. In the remaining sections we study the small dissipation case numerically.

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2. Existence of Stationary Solutions and Preliminary Estimates

We start by obtaining a simple bound on the L^2 norm of the solution. By multiplying the u-equation with $\overline{u}$ and integrating on $\mathbb{T}$ and then taking the imaginary part, we obtain

$$\frac{d}{dt}\|u\|_2^2 + 2\gamma\|u\|_2^2 = 2\Im\int f\overline{u}.$$

This implies by Gronwall's and Cauchy-Schwarz inequalities that for sufficiently large t (depending only on the L^2 norm of the initial data), we have

$$\|u\|_2 \leq 2\frac{\|f\|_2}{\gamma}. \tag{2}$$

We now study the stationary solutions of the system (1). Recall that n is real, and throughout the paper we assume that n and n_t are mean zero. Let (v,m) be a stationary solution of (1). Taking $u_t = n_t = n_{tt} = 0$ leads to

$$\begin{cases} v_{xx} + i\gamma v = mv + f, \quad x \in \mathbb{T}, \\ -m_{xx} = (|v|^2)_{xx}, \end{cases} \tag{3}$$

The second line of (3) implies that $m = -|v|^2 + ax + b$. Therefore, the periodicity and the mean zero assumption lead to $m = -|v|^2 + \frac{1}{2\pi}\|v\|_2^2$. Substituting to the first equation, it suffices to study

$$\Big[\frac{\partial^2}{\partial x^2} + i\gamma - \frac{1}{2\pi}\|v\|_2^2 + |v|^2\Big]v = f, \quad x \in \mathbb{T}. \tag{4}$$

Lemma 2.1. *Fix $f \in L^2$ and $\gamma > 0$. Any solution v of (4) satisfies the following a priori estimates*

$$\|v\|_2 \leq \frac{1}{\gamma}\|f\|_2, \tag{5}$$

$$\|v_x\|_2 \leq C\max(\gamma^{-3}\|f\|_2^3, \gamma^{-2}\|f\|_2^2, \gamma^{-1/2}\|f\|_2). \tag{6}$$

Proof. By multiplying (4) with $\overline{v}$ and integrating on $\mathbb{T}$ and then taking the imaginary part of the equation we obtain that

$$\gamma\|v\|_2^2 = \Im\int_{\mathbb{T}} f\overline{v}dx.$$

This implies (5) by Cauchy-Schwarz inequality.

On the other hand, taking the real part we obtain

$$\int |v_x|^2dx = \|v\|_{L^4}^4 - \frac{1}{2\pi}\|v\|_2^4 - \Re\int f\overline{v}dx.$$

By the Gagliardo-Nirenberg and Cauchy-Schwarz inequalities, we have

$$\|v_x\|_2^2 \le C\big(\|v_x\|_2\|v\|_2^3 + \|v\|_2^4 + \|f\|_2\|v\|_2\big).$$

This and (5) imply (6). $\square$

We now prove the existence of an H^1 solution v of (4) for large γ and/or small $\|f\|_{H^1}$ which is unique in a fixed ball in H^1. Uniqueness in the whole space will follow from Theorem 3.1 below.

Proposition 2.2. *Given $f \in H^1(\mathbb{T})$ if $\gamma > 0$ is sufficiently large, or for given $\gamma > 0$ if $\|f\|_{H^1}$ is sufficiently small, then we have a unique solution of (4) in the ball $B := \{v : \|v\|_{H^1} \le \frac{2\|f\|_{H^1}}{\gamma}\}$. Moreover $v \in H^3(\mathbb{T})$.*

Proof. First note that by Kato-Rellich theorem the operator $\frac{\partial^2}{\partial x^2} - \frac{1}{2\pi}\|v\|_2^2 + |v|^2$ is self adjoint on $L^2(\mathbb{T})$ for $v \in L^\infty(\mathbb{T}) \subset H^1(\mathbb{T})$. Therefore the operator

$$R_{\gamma,v} := \frac{\partial^2}{\partial x^2} - \frac{1}{2\pi}\|v\|_2^2 + |v|^2 + i\gamma$$

is invertible on $L^2(\mathbb{T})$ and we have

$$\|R_{\gamma,v}^{-1}\|_{L^2\to L^2} \le \frac{1}{\gamma}. \tag{7}$$

Let

$$T_{\gamma,f}(v) := R_{\gamma,v}^{-1} f.$$

It suffices to prove that $T_{\gamma,f}$ has a fixed point in H^1. To do that we will prove that $T_{\gamma,f}$ is a contraction on the ball B.

By the resolvent identity,

$$S^{-1} - T^{-1} = S^{-1}(T-S)T^{-1},$$

we have

$$R_{\gamma,v}^{-1} f = \big(\frac{\partial^2}{\partial x^2} + i\gamma\big)^{-1} f - \big(\frac{\partial^2}{\partial x^2} + i\gamma\big)^{-1}\big(-\frac{1}{2\pi}\|v\|_2^2 + |v|^2\big)R_{\gamma,v}^{-1} f.$$

Therefore, we obtain

$$\begin{aligned}\|R_{\gamma,v}^{-1} f\|_{H^1} &\le \frac{\|f\|_{H^1}}{\gamma} + C\langle\gamma^{-1/2}\rangle\gamma^{-1/2}\big\|\big(-\frac{1}{2\pi}\|v\|_2^2 + |v|^2\big)R_{\gamma,v}^{-1} f\big\|_{L^2} \\ &\le \frac{\|f\|_{H^1}}{\gamma} + C\langle\gamma^{-1/2}\rangle\gamma^{-1/2}\big\| - \frac{1}{2\pi}\|v\|_2^2 + |v|^2\big\|_{L^\infty}\big\|R_{\gamma,v}^{-1} f\big\|_{L^2}\end{aligned} \tag{8}$$

$$\leq \frac{\|f\|_{H^1}}{\gamma} + C\frac{\langle\gamma^{-1/2}\rangle}{\gamma^{3/2}}\|v\|_{H^1}^2\|f\|_2 \leq \frac{\|f\|_{H^1}}{\gamma}\big(1 + C\frac{\langle\gamma^{-1/2}\rangle}{\gamma^{1/2}}\|v\|_{H^1}^2\big).$$

Note that in the second to last inequality we used (7). Let $M = \frac{2}{\gamma}\|f\|_{H^1}$. The inequality above implies that for sufficiently large γ or for sufficiently small $\|f\|_{H^1}$, $T_{\gamma,f}$ maps $B = \{v \in H^1 : \|v\|_{H^1} \leq M\}$ into itself. Thus it suffices to prove that $T_{\gamma,f}$ is a contraction. Again by the resolvent identity and (8), we have (for $u, v \in B$)

$$\begin{aligned}
\|R_{\gamma,u}^{-1}f - R_{\gamma,v}^{-1}f\|_{H^1} &= \Big\|R_{\gamma,u}^{-1}\big(|v|^2 - |u|^2 + \frac{\|u\|_2^2 - \|v\|_2^2}{2\pi}\big)R_{\gamma,v}^{-1}f\Big\|_{H^1} \\
&\leq \frac{\|\big(|v|^2 - |u|^2 + \frac{\|u\|_2^2-\|v\|_2^2}{2\pi}\big)R_{\gamma,v}^{-1}f\|_{H^1}}{\gamma}\big(1 + C\frac{\langle\gamma^{-1/2}\rangle}{\gamma^{1/2}}\|u\|_{H^1}^2\big) \\
&\leq \Big\||v|^2 - |u|^2 + \frac{\|u\|_2^2 - \|v\|_2^2}{2\pi}\Big\|_{H^1}\frac{\|f\|_{H^1}}{\gamma^2} \\
&\quad\times \big(1 + C\frac{\langle\gamma^{-1/2}\rangle}{\gamma^{1/2}}\|v\|_{H^1}^2\big)\big(1 + C\frac{\langle\gamma^{-1/2}\rangle}{\gamma^{1/2}}\|u\|_{H^1}^2\big) \\
&\leq CM\big(1 + C\frac{\langle\gamma^{-1/2}\rangle}{\gamma^{1/2}}M^2\big)^2\frac{\|f\|_{H^1}}{\gamma^2}\|u - v\|_{H^1}.
\end{aligned}$$

Therefore $T_{\gamma,f}$ is a contraction on B for small $\|f\|_{H^1}$ or large γ. Finally by the following calculation the fix point $v \in B$ is in $H^3(\mathbb{T})$,

$$\begin{aligned}
(9)\qquad \|v\|_{H^3} = \|T_{\gamma,f}(v)\|_{H^3} &\leq \langle\gamma^{-1}\rangle\|f\|_{H^1} + \langle\gamma^{-1}\rangle\Big\| - \frac{1}{2\pi}\|v\|_2^2 + |v|^2\Big\|_{H^1}\|R_{\gamma,v}^{-1}f\|_{H^1} \\
&\leq \langle\gamma^{-1}\rangle\|f\|_{H^1} + C\langle\gamma^{-1}\rangle\|f\|_{H^1}^3\gamma^{-3},
\end{aligned}$$

using the standard elliptic estimate $\|\big(\frac{\partial^2}{\partial x^2} + i\gamma\big)^{-1}f\|_{H^3} \leq \langle\gamma^{-1}\rangle\|f\|_{H^1}$. □

3. Attractor in the Case of Large Dissipation

Recall that the energy space is $X = H^1 \times L^2 \times \dot{H}^{-1}$. We will prove under some conditions on $\gamma, \delta, \|f\|_{H^1}$ that all solutions of (1) converge to the stationary solution $(v, -|v|^2 + \frac{1}{2\pi}\|v\|_2^2, 0)$ in X as $t \to \infty$. This also implies the uniqueness of the stationary solution v under these conditions.

Theorem 3.1. *Given $\|f\|_{H^1}$ and $\delta > 0$, the following statement holds if γ is sufficiently large. Consider $(u(0), n(0), n_t(0)) \in X$ where $n(0)$ and $n_t(0)$ are mean-zero. Then, the solution (u, n, n_t) of (1) converges to the stationary solution $(v, -|v|^2 + \frac{1}{2\pi}\|v\|_2^2, 0)$ in X as $t \to \infty$.*

Proof. Given solution (u, n, n_t) of (1), let

$$(w, z, z_t) = (u - v, n + |v|^2 - \frac{1}{2\pi}\|v\|_2^2, n_t).$$

Note that z and z_t are mean-zero. The equation for (w, z, z_t) is the following

$$\begin{cases} iw_t + w_{xx} + i\gamma w = z(w+v) - |v|^2 w + \frac{1}{2\pi}\|v\|_2^2 w, \quad x \in \mathbb{T}, \quad t \in [0,\infty), \\ z_{tt} - z_{xx} + \delta z_t = (|w+v|^2 - |v|^2)_{xx}. \end{cases} \tag{10}$$

Fix $\epsilon > 0$ and let

$$H = \big\|\partial_x^{-1}(z_t + \epsilon z)\big\|_2^2 + \|z\|_2^2 + 2\|w_x\|_2^2 + 2\int_{\mathbb{T}} z(|w+v|^2 - |v|^2) + \|w\|_2^2.$$

The above quantity H was introduced in [6] to obtain bounds in the energy space. We note that H is bounded by a constant multiple of the energy norm for any fixed ϵ.

We have

$$\begin{aligned}
\frac{d}{dt}\big\|\partial_x^{-1}(z_t + \epsilon z)\big\|_2^2 &= 2\int \partial_x^{-1}(z_t+\epsilon z)\partial_x^{-1}(z_{tt} + \epsilon z_t) \\
&= 2\int \partial_x^{-1}(z_t+\epsilon z)\partial_x^{-1}[(z + |w+v|^2 - |v|^2)_{xx} + (\epsilon - \delta)z_t] \\
&= -2\int (z_t + \epsilon z)(z + |w+v|^2 - |v|^2) - 2(\delta - \epsilon)\int \partial_x^{-1}(z_t+\epsilon z)\partial_x^{-1} z_t \\
&= -\frac{d}{dt}\|z\|_2^2 - 2\epsilon\|z\|_2^2 - 2\int z_t(|w+v|^2 - |v|^2) - 2\epsilon\int z(|w+v|^2 - |v|^2) \\
&\quad - 2(\delta - \epsilon)\|\partial_x^{-1} z_t\|_2^2 - 2\epsilon(\delta - \epsilon)\int \partial_x^{-1} z \partial_x^{-1} z_t.
\end{aligned}$$

Using

$$\big\|\partial_x^{-1}(z_t + \epsilon z)\big\|_2^2 = \big\|\partial_x^{-1} z_t\big\|_2^2 + \epsilon^2\|\partial_x^{-1} z\|_2^2 + 2\epsilon\int \partial_x^{-1} z \partial_x^{-1} z_t,$$

we obtain the following energy-type identity

$$\begin{aligned}
\frac{d}{dt}\big\|\partial_x^{-1}(z_t + \epsilon z)\big\|_2^2 + \frac{d}{dt}\|z\|_2^2 &= -2\epsilon\|z\|_2^2 - 2\int z_t(|w+v|^2 - |v|^2) \\
&\quad - 2\epsilon\int z(|w+v|^2 - |v|^2) - (\delta - \epsilon)\|\partial_x^{-1} z_t\|_2^2 \\
&\quad - (\delta - \epsilon)\big\|\partial_x^{-1}(z_t + \epsilon z)\big\|_2^2 + (\delta - \epsilon)\epsilon^2\|\partial_x^{-1} z\|_2^2.
\end{aligned} \tag{11}$$

Now consider the derivative of the remaining terms in the definition of H:

$$2\frac{d}{dt}\|w_x\|_2^2 = -4\Re\int \overline{w_{xx}} w_t = -4\gamma\|w_x\|_2^2 - 4\Im\int \overline{w_{xx}}\big[z(w+v) - |v|^2 w\big], \tag{12}$$

and

$$2\frac{d}{dt}\int z(|w+v|^2-|v|^2) = 2\int z_t(|w+v|^2-|v|^2) + 4\Re\int z\overline{w_t}(w+v) \tag{13}$$
$$= 2\int z_t(|w+v|^2-|v|^2) + 4\Im\int z\overline{w_{xx}}(w+v)$$
$$-4\gamma\Re\int z\overline{w}(w+v) + 4\Im\int zv\overline{w}\big[|v|^2 - \frac{1}{2\pi}\|v\|_2^2\big].$$

We also have

$$\partial_t\|w\|_2^2 = -2\gamma\|w\|_2^2 + 2\Im\int z\overline{w}v. \tag{14}$$

Combining (11)-(14), we observe

$$\frac{d}{dt}H = -2\epsilon\|z\|_2^2 - 2\epsilon\int z(|w+v|^2-|v|^2) - (\delta-\epsilon)\|\partial_x^{-1}z_t\|_2^2 - (\delta-\epsilon)\big\|\partial_x^{-1}(z_t+\epsilon z)\big\|_2^2$$
$$+(\delta-\epsilon)\epsilon^2\|\partial_x^{-1}z\|_2^2 - 4\gamma\|w_x\|_2^2 + 4\Im\int w\overline{w_{xx}}|v|^2 - 4\gamma\Re\int z\overline{w}(w+v)$$
$$+4\Im\int zv\overline{w}\big[|v|^2-\frac{1}{2\pi}\|v\|_2^2\big] - 2\gamma\|w\|_2^2 + 2\Im\int z\overline{w}v.$$

Hence,

$$\frac{d}{dt}H = -\epsilon H - \epsilon\|z\|_2^2 - (\delta-\epsilon)\|\partial_x^{-1}z_t\|_2^2 - (\delta-2\epsilon)\big\|\partial_x^{-1}(z_t+\epsilon z)\big\|_2^2 - (2\gamma-\epsilon)\|w\|_2^2$$
$$+2\Im\int z\overline{w}v + (\delta-\epsilon)\epsilon^2\|\partial_x^{-1}z\|_2^2 - (4\gamma-2\epsilon)\|w_x\|_2^2 - 4\Im\int w\overline{w_x}(|v|^2)_x$$
$$-4\gamma\Re\int z\overline{w}(w+v) + 4\Im\int zv\overline{w}\big[|v|^2-\frac{1}{2\pi}\|v\|_2^2\big].$$

Let $\epsilon = \min(\frac{1}{2\delta}, \frac{\delta}{2}, \gamma)$. Since z is mean-zero, the choice of ϵ implies

$$(\delta-\epsilon)\epsilon^2\|\partial_x^{-1}z\|_2^2 \le \frac{\epsilon}{2}\|z\|_2^2.$$

Therefore, we have

$$\frac{d}{dt}H \le -\epsilon H - \frac{\epsilon}{2}\|z\|_2^2 - \gamma\|w\|_2^2 - 2\gamma\|w_x\|_2^2 + 2\Big|\int z\overline{w}v\Big| + 4\Big|\int w\overline{w_x}(|v|^2)_x\Big|$$
$$+4\gamma\Big|\int z\overline{w}(w+v)\Big| + 4\Big|\int zv\overline{w}\Big[|v|^2-\frac{1}{2\pi}\|v\|_2^2\Big]\Big|$$
$$\le -\epsilon H - \frac{\epsilon}{2}\|z\|_2^2 - \gamma\|w\|_2^2 - 2\gamma\|w_x\|_2^2$$
$$+C\Big[\|z\|_2\|w\|_2\|v\|_{H^1} + \|w\|_2\|w_x\|_2\|v\|_{H^2}^2 + \|z\|_2\|w\|_2\|v\|_{H^1}^3$$
$$+\gamma\|z\|_2\|w\|_{H^1}\|w+v\|_2\Big]$$

$$
\begin{aligned}
&= -\epsilon H - \frac{\epsilon}{2}\|z\|_2^2 - \gamma\|w\|_2^2 - 2\gamma\|w_x\|_2^2 \\
&\quad + \big[\mathcal{I} + \mathcal{II} + \mathcal{III} + \mathcal{IV}\big].
\end{aligned}
$$

Note that by (2) we have

$$\|w+v\|_2 \leq 2\frac{\|f\|_2}{\gamma}$$

for sufficiently large t. Using this, we can bound term $\mathcal{IV}$ by

$$
\begin{aligned}
2C\|f\|_2\|z\|_2\|w\|_{H^1} &\leq \frac{\epsilon}{10}\|z\|_2^2 + \frac{C_1\|f\|_2^2}{\epsilon}\|w\|_{H^1}^2 \\
&\leq \frac{\epsilon}{10}\|z\|_2^2 + \frac{C_1\|f\|_2^2}{\epsilon}\|w\|_2^2 + \frac{C_1\|f\|_2^2}{\epsilon}\|w_x\|_2^2 \leq \frac{\epsilon}{10}\|z\|_2^2 + \frac{\gamma}{10}\|w\|_2^2 + \frac{\gamma}{10}\|w_x\|_2^2,
\end{aligned}
$$

provided that $\epsilon\gamma \gg \|f\|_2^2$. Summands $\mathcal{I} - \mathcal{III}$ can be bounded by the same right hand side provided that

$$\epsilon\gamma \gg \|v\|_{H^1}^2 + \|v\|_{H^1}^6, \quad \text{and} \quad \gamma \gg \|v\|_{H^2}^2.$$

By the estimates on v, we see that for fixed δ and $\|f\|_{H^1}$, if γ is sufficiently large, we have for sufficiently large t,

$$\frac{d}{dt}H \leq -\epsilon H.$$

This implies that H goes to zero as $t \to \infty$.

Observe that

$$\Big|\int_{\mathbb{T}} z(|w+v|^2 - |v|^2)\Big| \leq C\|z\|_2\|w\|_{H^1}(\|w\|_2 + \|v\|_2),$$

and that, by (2) and (5), (for large γ and t) we have $\|w\|_2 + \|v\|_2 \ll 1$. Therefore, we have

$$H \geq C\big(\|z_t\|_{H^{-1}}^2 + \|z\|_2^2 + \|w\|_{H^1}^2\big).$$

This completes the proof. $\square$

4. Numerical Methods

In this section we will study the Schrödinger-Dirac model (see, e.g., [5]) numerically. In particular, we apply the time-splitting method of [8] as applied to the soliton dynamics in,

for instance, [9]. Let us recall the equivalent system to (1) derived in [5], which are

$$\begin{cases} (i\partial_t + \partial_x^2 + i\gamma)u = nu + f, \\ (i\partial_t - d + i\delta)n = d(|u|^2), \\ (u(x,0), n(x,0)) = (u_0, n_0) \in H^1 \times L^2, \end{cases} \tag{15}$$

where $d = (-\partial_{xx})^{\frac{1}{2}}$. Note, this can be re-written as

$$\partial_t \begin{pmatrix} u \\ n \end{pmatrix} = L \begin{pmatrix} u \\ n \end{pmatrix} + N(u, n, f),$$

where

$$L = \begin{bmatrix} i\partial_x^2 - \gamma & 0 \\ 0 & -id - \delta \end{bmatrix}$$

and

$$N = \begin{pmatrix} -inu - if \\ -id(|u|^2) \end{pmatrix}.$$

The algorithm takes place as a pseudospectral method on the Fourier side, though it implements integrating factor, time-splitting, fourth-order Runge-Kutta schemes and contour integration all at once. The key idea is to look at the evolution over a time step, h, as the integral

$$\begin{pmatrix} u_{m+1} \\ n_{m+1} \end{pmatrix} = e^{Lh} \begin{pmatrix} u_m \\ n_m \end{pmatrix} + e^{Lh} \int_0^h e^{-Ls} N(u(t_m + s), n(t_m + s), f(t_m + s)) ds,$$

which can be approximated using a Runge-Kutta method (see Cox-Matthews [4]) as

$$\begin{aligned} \begin{pmatrix} u_{m+1} \\ n_{m+1} \end{pmatrix} &= e^{Lh} \begin{pmatrix} u_m \\ n_m \end{pmatrix} + h^{-2} L^{-3} \times \\ &\Big(\Big[-4 - Lh + e^{Lh}(4 - 3Lh + (Lh)^2) \Big] N(u_m, n_m, f(t_m)) + \\ &+ 2 \Big[2 + Lh + e^{Lh}(-2 + Lh) \Big] \big(N(a_{m,1}, a_{m,2}, f(t_m + h/2)) \\ &\qquad + N(b_{m,1}, b_{m,2}, f(t_m + h/2)) \big) \\ &+ \Big[-4 - 3h - (Lh)^2 + e^{Lh}(4 - Lh) \Big] N(c_{m,1}, c_{m,2}, f(t_m + h)) \Big), \end{aligned}$$

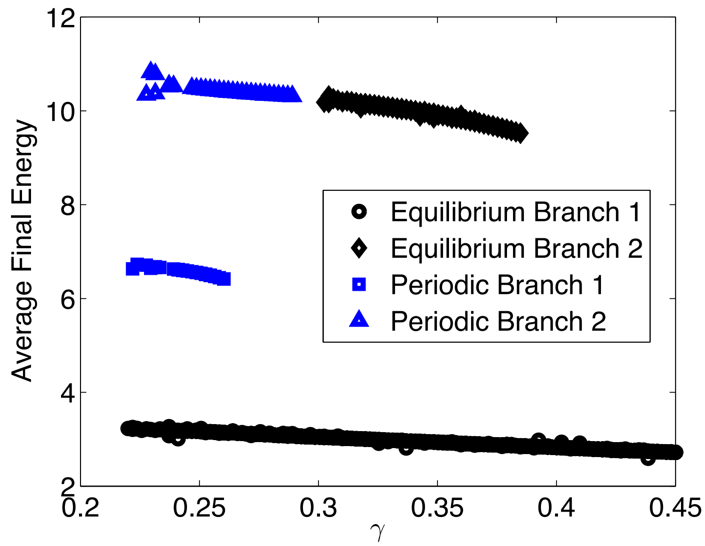

FIGURE 1. A plot of periodic and converged orbits for $\gamma = .2$ to $\gamma = .45$ with $\eta = 1.0$ versus the final energy of two distinct branches of converged solutions and measured over the average final energy of one period of two distinct periodic branches. The solutions were plotted over a choice of $A = -7.0, -4.05, -3.07, -1.0, 0.0, 1.0, 1.95, 3.97, 10.0$.

where

$$\begin{aligned} a_m &= e^{Lh/2}\begin{pmatrix} u_m \\ n_m \end{pmatrix} + L^{-1}(e^{Lh/2} - Id)N(u_m, n_m, f(t_m)), \\ b_m &= e^{Lh/2}\begin{pmatrix} u_m \\ n_m \end{pmatrix} + L^{-1}(e^{Lh/2} - Id)N(a_{m,1}, a_{m,2}, f(t_m + h/2)), \\ c_m &= e^{Lh/2}a_m + L^{-1}(e^{Lh/2} - Id)(2N(b_{m,1}, b_{m,2}, f(t_m + h/2)) - N(u_m, n_m, f(t_m)). \end{aligned}$$

However, such an algorithm can have problems if L has eigenvalues near 0. To avoid such problems the algorithm is slightly modified by evaluating contour integrals over whole discs, which are approximated by appropriate Riemann sums.

5. Numerical Results in the Case of Small Dissipation

To begin, in an attempt to model the non-trivial dynamics in the Zakharov system, we follow some of the ideas in [3] to analyze a series of numerically integrated solutions of

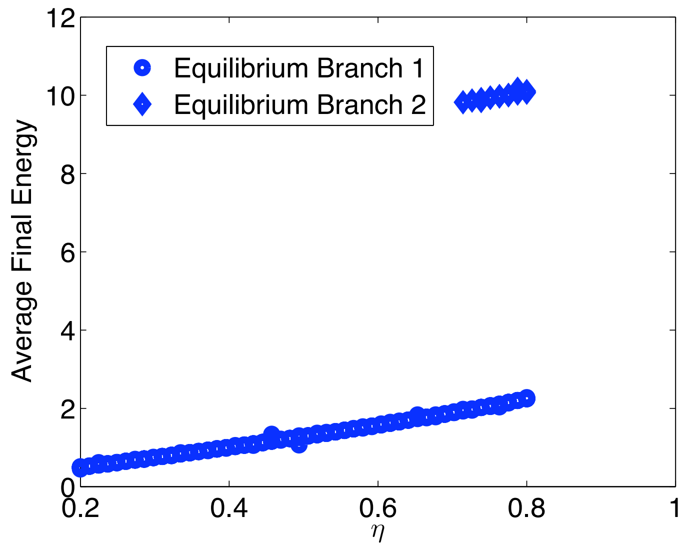

FIGURE 2. A plot of a range of η from .2 to .8 versus the measured average final energy blown-up near a bifurcation point for $\gamma = .225$. The solutions were plotted over a choice of $A = -4.05, -3.07, -1.0, 0.0, 1.0, 4.0$.

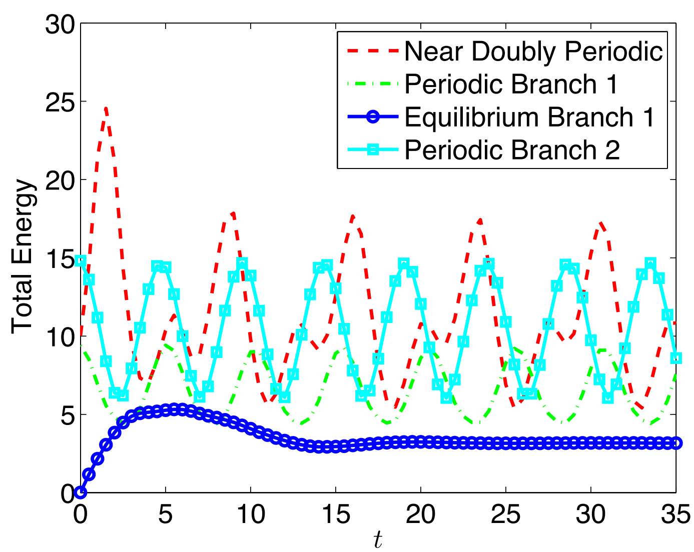

FIGURE 3. Plot of the Energy versus (appropriately translated) time for initial $A = -4.05$ (Nearly Doubly Periodic), -4.0 (Periodic Branch 1), -5.0 (Periodic Branch 2), 0.0 (Equilibrium Branch 1) for $\gamma = .25$ and $\eta = 1.0$.

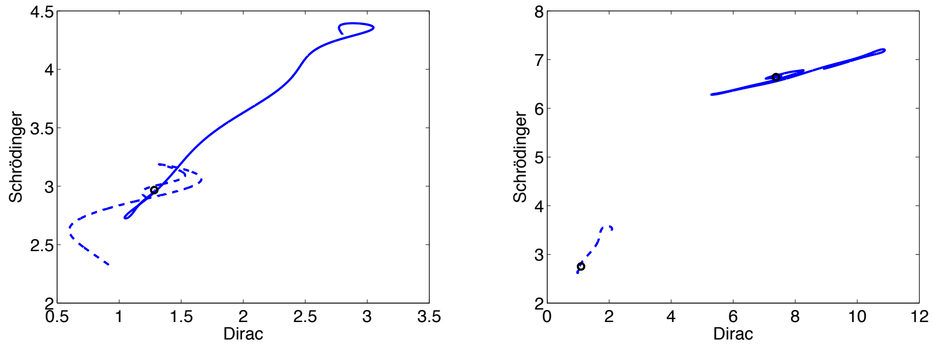

FIGURE 4. Plots of the Dirac Final Energy versus Schrödinger Final Energy for two solutions that converge to an equilibrium point of the dynamics with $\gamma = .25$, $\eta = 1.0$ (left, Equilibrium Branch 1) and $\gamma = .35$, $\eta = 1.0$ (right, Equilibrium Branches 1, 2). Specifically, we take $A = 0.0, -1.0$ and $A = -4.0, 0.0$ respectively and plot the parametric evolution from $t = 20.0$ to $t = 40.0$. Note that the trajectories converge to the same equilibrium point for $\gamma = .25$ and to different equilibrium points for $\gamma = .35$.

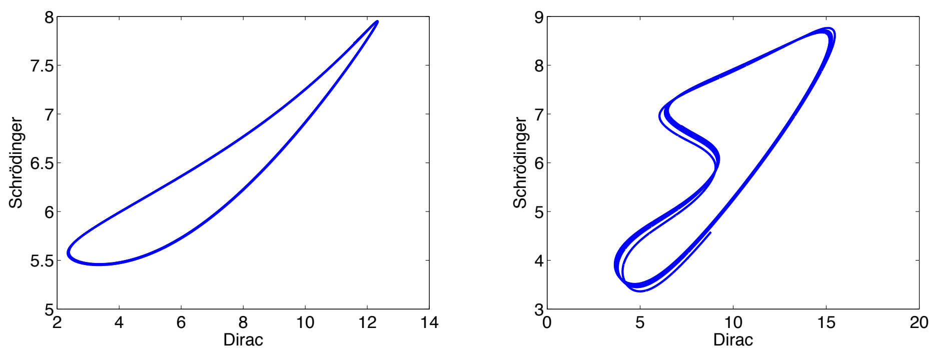

FIGURE 5. Plots of the Dirac Final Energy versus Schrödinger Final Energy in the case of periodic orbits in the dynamics with $\gamma = .25$, $\eta = 1.0$ and $A = -5.0$ (left, Periodic Branch 2), $A = -4.05$ (right, Near Doubly Periodic). We plot the parametric evolution from $t = 20.0$ to $t = 40.0$. The solution on the right will, along upon further evolution, eventually converge to Periodic Branch 1 as in Figure 3. The doubly periodic dynamics occur only for a very small window of A values, $A \in [-4.05, -4]$.

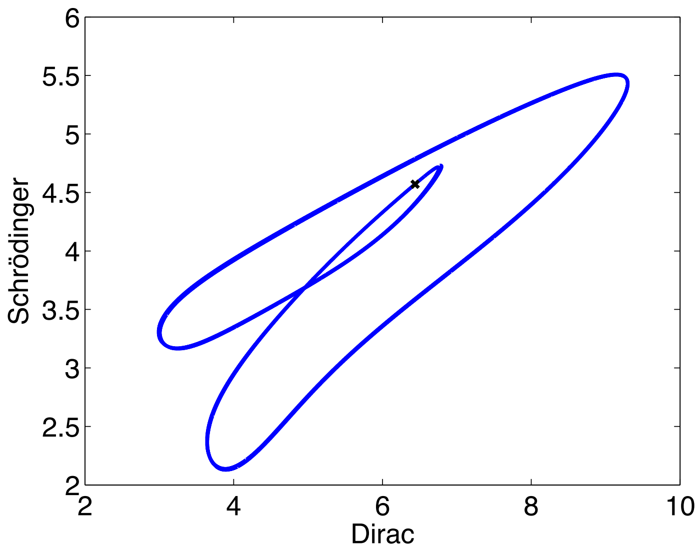

FIGURE 6. Plot of the Dirac Final Energy versus Schrödinger Final Energy in the case of a stable doubly periodic orbit from Periodic Branch 2 corresponding to $\gamma = .225$, $\eta = 1.0$, and $A = 1.9005$. We plot over the parametric evolution over two full orbits from $t = 180.0$ to $t = 200.0$. The doubly periodic dynamics occur only for a very small window of A values, $A \in [1.9, 2]$.

(15). In our numerical experiments we observe a great deal of energy exchange between the Schrödinger and Dirac solutions, hence we will focus on relatively small energy initial data in order to justify that our numerics are valid on long time scales. If the Fourier modes become too large at the edges of the spectrum, we do not consider the solution to be appropriately accurate, hence all simulations included here will have small contributions at high frequency. The time scale on which we integrate is generally $T = 50.0$ with the time step $h \sim 1e - 4$. We will begin by taking 32 Fourier modes on which to evolve. In addition, our contour integrals in the numerical evaluation of L^{-1} will be taken as a mean of 64 equidistributed points along the disc.

We look for solutions with

$$u(0, x) = A \sin(x), \quad n(0, x) = 0, \quad \delta = \gamma, \quad f = \eta \sin(x),$$

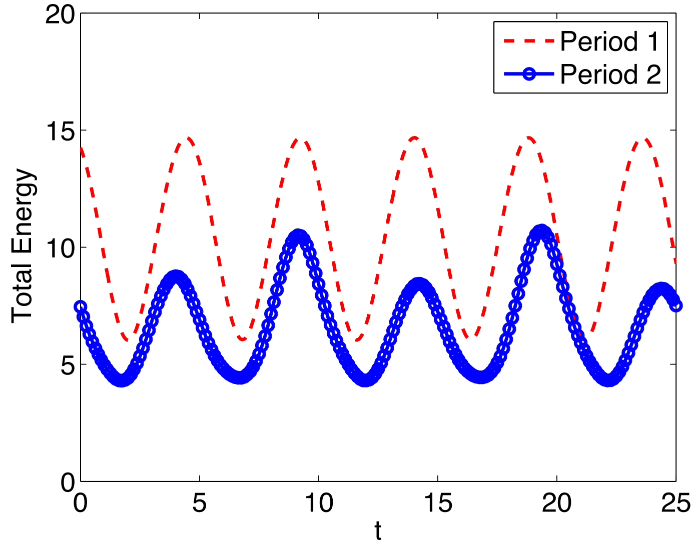

FIGURE 7. Plots of the Energy versus (appropriately translated) time for periodic doubling along Periodic Branch 2 of solutions with γ moving from .25 (Period 1) to .225 (Period 2) with $A = -5.0$ and $A = 1.90$ respectively.

where we vary

$$-10 < A < 10, \quad .2 < \gamma < 1.0, \quad .2 < \eta \leq 1.0. \tag{16}$$

For the range of η in (16), given $\gamma > 1$, we observe that the dynamics tend to a fixed equilibrium solution for a large set of A values, putting us solidly in the dynamics of Section 3. However, for γ much smaller, we observe much richer dynamics in the phase space, particularly in the form of quasiperiodic orbits and equilibrium solutions. We use the phrase quasiperiodic orbits to mean that the orbits are periodic up to preset numerical integration parameters. Namely, we assume that the solution returns to a previous configuration within an error of at most $5e-3$. However, many such orbits can then be integrated over many periods and are in that sense quite numerically stable.

As one may expect, the nature of our orbits can change as we vary γ or η, which accounts for movement in the bifurcation diagram presented in Figures 1 and 2. The other figures present particular solutions for $\eta = 1$ and for various values of γ and A justifying the bifurcation diagram in Figure 1. In Figure 3, we present three solutions corresponding to periodic branches and the equilibrium branch from Figure 1 for $\gamma = .25$, and an unstable

near doubly periodic solution that eventually converges to the solution in periodic branch 1 (also see Figures 5 and 6). In Figure 4, we present sample solutions corresponding to equilibrium branches. For $\gamma = 0.25$, we have one equilibrium solution and for $\gamma = 0.35$, there are two distinct equilibrium solutions. Finally in Figure 7, we present period doubling by plotting the time behavior of two limiting solutions from Periodic Branch 2 with γ moving from 0.25 to 0.225.

6. Discussion

We have analytically and numerically observed rich dynamics in the dissipative periodic Zakharov system with forcing. Open problems for future consideration include understanding the large exchange of energy from Schrödinger to Dirac, classifying dynamics for a larger range of energies, finding more bifurcation points, etc. For small γ values, we observe numerically that there is a great deal of energy transfer from the Schrödinger equation into the Dirac equation at the outset of the dynamics. As a result, it is a challenge to probe extremely large energies within the current numerical framework. In addition, such energy transfers seem to dissipate through the damping on incredibly slow scales, making it difficult to search for meaningful dynamics such as further period doubling, Hopf bifurcation branches and more as the orbits behave do not settle onto a fixed point or a stable orbit for very long time scales. However, we mention that using the orbits we have shadowed here, it is likely that a variation of the gradient methods in [1] could allow one to construct nearby periodic solutions with great accuracy and hence move along the solution branches in a more robust manner.

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